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Classical Methods of Structural Analysis

Louis F. Geschwindner

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Session Description
20.4 Moment Areas and Elastic Weights
July 8, 2019

This lesson will continue the discussion of deflection calculations and the principles upon which the various available methods are based. It will include discussion of direct integration, elastic weights, moment area, and conjugate beam. Slopes and deflections will be calculated at specific points on a beam span and slope and moment diagrams will be sketched. The use of the conjugate beam method for determining redundant moments will be introduced.

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Learning Objectives:

- Explain the direct integration method for determining deflections.
- Describe the moment area method for deflection calculations
- Calculate slopes and deflections at specific points on a beam span.
- Describe the conjugate beam method for determining redundant moments.

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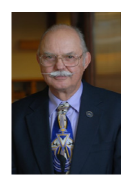
Classical Methods of Structural Analysis

Session 4: Moment Areas and Elastic Weights
July 8, 2019



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Classical Methods of Structural Analysis:
How we did it before computers

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Lesson 4
Moment Area and Elastic Weights

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Lesson 4
Moment Area and Elastic Weights

- Develop "classical beam theory"
- Apply the double integration approach for calculating flexural slopes and deflections.
- Extend the double integration method to the moment area method for slope and deflection calculations.
- Introduce the method of elastic weights and extend it to the better known and more useful conjugate beam method.
- Consider how varying moment of inertia will influence slope and deflection.

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Deflections by Work

- One issue with the Work methods discussed over the last 2 lessons (Real Work and Virtual Work) is the limitation that only one displacement or rotation can be determined at a time.
- To determine the deflected shape, a unit virtual load must be applied at a series of locations along the member and the resulting deflections plotted.
- However, there are methods to directly determine the deflected shape and rotations for the entire beam which we will investigate in this lesson.

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Basic Beam Theory

- In Lesson 1 we used equilibrium to develop the relationship between shear and moment diagrams.
- To go further and calculate rotations and deflections, we must look to other considerations.
- The geometry of the bent beam and the properties of the material will be needed.

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Basic Beam Theory

- In Lesson 2 we accepted that we knew the relationship between the stress on a cross section and the moment at that section on a beam.

$$f_y = \frac{My}{I}$$

- We also accepted Hooke's Law, the relationship between stress and strain.

$$E = \frac{f_y}{\varepsilon_y}$$

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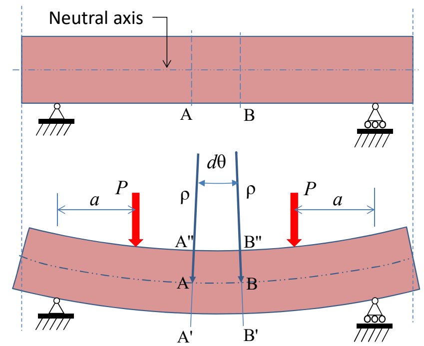
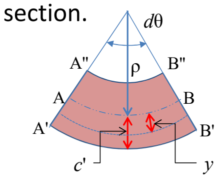
Basic Beam Theory

Now consider an elastic beam of any cross section.





1. Lines A"AA' and B"BB' will be straight lines.
2. After the beam deflects, arcs A'B' and AB will be very nearly circular arcs. The arc lengths can be determined as:

$$L_{AB} = \rho d\theta$$

$$L_{A'B'} = (\rho + c')d\theta$$

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Basic Beam Theory

- Since before loading lines L_{AB} and $L_{A'B'}$ were the same length, the difference in length is the elongation at c'. Thus,

$$L_{A'B'} - L_{AB} = (\rho + c')d\theta - \rho d\theta = c'd\theta$$

- and the strain at c' produced by the loading is.

$$\varepsilon_{c'} = \frac{L_{A'B'} - L_{AB}}{L_{AB}} = \frac{c'd\theta}{\rho d\theta} = \frac{c'}{\rho}$$

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Basic Beam Theory

- By a similar process, the strain at any location, y, from the neutral axis can be determined as:

$$\varepsilon_y = \frac{y}{\rho}$$

- Combining the flexure formula and Hooke's Law, we can solve for the strain.

$$f_y = \frac{My}{I}, \qquad E = \frac{f_y}{\varepsilon_y} \qquad \rightarrow \quad \varepsilon_y = \frac{My}{EI}$$

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Basic Beam Theory

- Setting these two equations for strain equal yields

$$\frac{y}{\rho} = \frac{My}{EI}$$

and solving for the radius of curvature yields:

$$\rho = \frac{EI}{M}$$

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Basic Beam Theory

- For a rectangular coordinate system, the radius of curvature is given by

$$\rho = \frac{\left(1 + (dy/dx)^2\right)^{1.5}}{d^2y/dx^2}$$

- For the majority of our problems, the slope of the deflection curve, dy/dx, is very small and when squared is negligible when compared to unity.

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Basic Beam Theory

- Thus, a good approximation for the radius of curvature with the numerator equal to 1.0 is

$$\rho = \frac{1}{d^2y/dx^2}$$

- Substituting this into the equation relating the radius of curvature to the moment, (slide 17)

$$\frac{1}{d^2y/dx^2} = \frac{EI}{M} \quad \text{which yields} \quad \frac{d^2y}{dx^2} = \frac{M}{EI}$$

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Basic Beam Theory

- Jacob Bernoulli made the significant discoveries that lead Leonhard Euler and Daniel Bernoulli to first put together a useful beam theory in about 1750. Thus, it is referred to as the Euler-Bernoulli beam theory.

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

- This is also known as “engineers beam theory” or “classical beam theory.”

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Basic Beam Theory

- Although presented in the mid 1700's, it was not put to practical use on large structures until the Eiffel Tower (1889) and the Ferris Wheel (1893).

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Basic Beam Theory

- Remember some of the major assumptions that were made to get this far.
  - A plane before bending remains a plane after bending – confirmed through many experiments.
  - Curvature of the deflected beam is a circular arc.
  - There is no interaction between shear and bending.
  - Slope of the deflected beam is sufficiently small that the square of the slope can be ignored when compared with unity.

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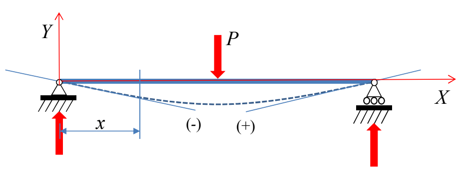
Basic Beam Theory

- Sign Convention, x-y coordinate system.



For the given axis system:
1. The load, P, is negative
2. At any location x, the moment is positive.
3. The rate of change of slope is positive.

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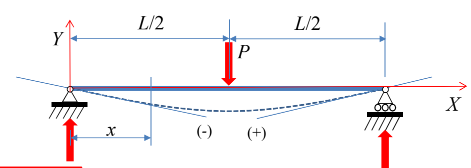
Double Integration

- Simple beam with a concentrated load at mid-span



For $0 \leq x \leq L/2$

$$M_x = \frac{P}{2}x \quad \text{thus} \quad \frac{d^2y}{dx^2} = \frac{M_x}{EI} = \frac{Px}{2EI}$$

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Double Integration

- Integrating once we have

$$\frac{dy}{dx}=\frac{1}{EI}\left[\frac{Px^2}{4}+C_1\right]$$

From symmetry we know that at $x = L/2$, the slope, $dy/dx = 0$. Thus, we can determine that

$$C_1=-\frac{PL^2}{16}$$

and

$$\frac{dy}{dx}=\frac{1}{EI}\left[\frac{Px^2}{4}-\frac{PL^2}{16}\right]$$

Remember this equation is only valid for $0 \le x \le L/2$

Double Integration

- Integrating a second time

$$y=\frac{1}{EI}\left[\frac{Px^3}{12}-\frac{PL^2x}{16}+C_2\right]$$

We know that at $x = 0$, the deflection, $y = 0$. Thus, we can determine that

$$C_2=0$$

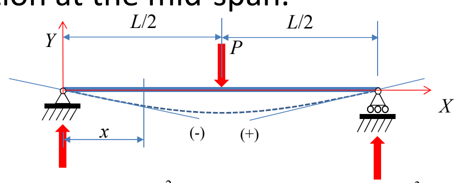
and

$$y=\frac{1}{EI}\left[\frac{Px^3}{12}-\frac{PL^2x}{16}\right]$$

Remember this equation is only valid for $0 \le x \le L/2$

Double Integration

- By inspection of the slope and deflection equations, we know that the maximum slope occurs at the support and the maximum deflection at the mid-span.



$$\frac{dy}{dx}=\theta_0=-\frac{PL^2}{16EI}$$

$$y=\Delta_{L/2}=-\frac{PL^3}{48EI}$$

Double Integration

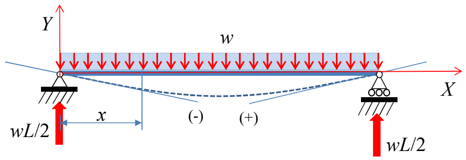
- We are now in a position to determine the slope and deflection at any point on the left half of the beam span, and by symmetry, the right half. At the quarter points, $x = L/4$

$$\theta_{L/4}=\frac{1}{EI}\left[\frac{Px^2}{4}-\frac{PL^2}{16}\right]=\frac{1}{EI}\left[\frac{PL^2}{64}-\frac{PL^2}{16}\right]=-\frac{3PL^2}{64EI}$$

$$\Delta_{L/4}=\frac{1}{EI}\left[\frac{Px^3}{12}-\frac{PL^2x}{16}\right]=\frac{1}{EI}\left[\frac{PL^3}{768}-\frac{PL^3}{64}\right]=-\frac{11PL^3}{768EI}$$

Double Integration

- Consider a simple beam with a uniform load



For $0 \le x \le L$

$$M_x = \frac{wLx}{2} - \frac{wx^2}{2}$$

$$\frac{d^2y}{dx^2} = \frac{M_x}{EI} = \frac{1}{EI}\left[\frac{wLx}{2} - \frac{wx^2}{2}\right]$$

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Double Integration

- Integrating once

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{wLx^2}{4} - \frac{wx^3}{6} + C_1\right]$$

From symmetry we know that at $x = L/2$, the slope, $dy/dx = 0$. Thus we can determine that

$$C_1 = -\frac{wL^3}{24}$$

and $$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{wLx^2}{4} - \frac{wx^3}{6} - \frac{wL^3}{24}\right]$$

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Double Integration

- Integrating a second time

$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24} + C_2\right]$$

We know that at $x = 0$, the deflection, $y = 0$. Thus, we can determine that

$$C_2 = 0$$

and $$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

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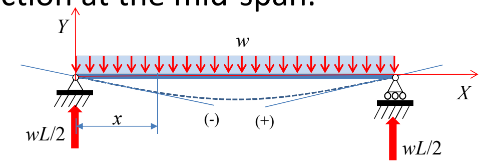
Double Integration

- By inspection of the slope and deflection equations, we know that the maximum slope occurs at the support and the maximum deflection at the mid-span.



$$\frac{dy}{dx} = \theta_0 = -\frac{wL^3}{24EI} \qquad y = \Delta_{L/2} = -\frac{5wL^4}{384EI}$$

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Double Integration

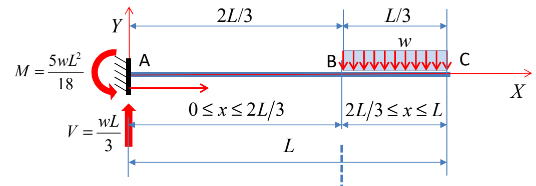
- We are now in a position to determine the slope and deflection at any point on the beam. At the quarter points, $x = L/4$

$$\theta_{L/4} = \frac{1}{EI}\left[\frac{wLx^2}{4} - \frac{wx^3}{6} - \frac{wL^3}{24}\right] = \frac{1}{EI}\left[\frac{wL^3}{64} - \frac{wL^3}{384} - \frac{wL^3}{24}\right] = -\frac{11wL^3}{384EI}$$

$$\Delta_{L/4} = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right] = \frac{1}{EI}\left[\frac{wL^4}{768} - \frac{wL^4}{6144} - \frac{wL^4}{96}\right] = -\frac{57wL^4}{6144EI}$$

Double Integration

- Consider a beam that is not symmetric.



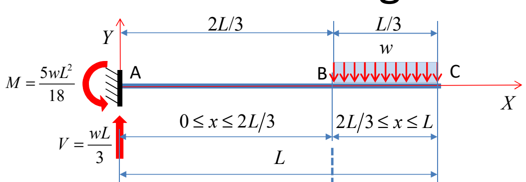
$$M_x = \frac{wLx}{3} - \frac{5wL^2}{18}$$

$$M_x = \frac{wLx}{3} - \frac{5wL^2}{18} - w(x - 2L/3)\left(\frac{x - 2L/3}{2}\right)$$

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{wLx^2}{6} - \frac{5wL^2x}{18} + C_1\right]$$

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{wLx^2}{6} - \frac{5wL^2x}{18} - \frac{wx^3}{6} + \frac{wLx^2}{3} - \frac{2wL^2x}{9} + C_3\right]$$

Double Integration



$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{wLx^2}{6} - \frac{5wL^2x}{18} + C_1\right]$$

at $x = 0$, $\frac{dy}{dx} = 0$

therefore $C_1 = 0$ and

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{wLx^2}{6} - \frac{5wL^2x}{18}\right]$$

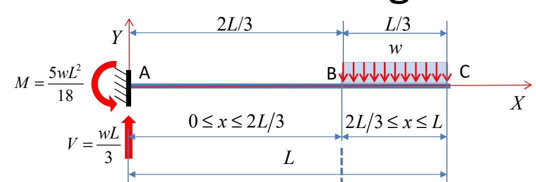
$$\frac{dy}{dx} = \frac{1}{EI}\left[-\frac{wx^3}{6} + \frac{wLx^2}{2} - \frac{wL^2x}{2} + C_3\right]$$

at $x = 2L/3$, $\left(\frac{dy}{dx}\right)_{\text{Left}} = \left(\frac{dy}{dx}\right)_{\text{Right}}$

therefore $C_3 = \frac{4wL^3}{81}$ and

$$\frac{dy}{dx} = \frac{1}{EI}\left[-\frac{wx^3}{6} + \frac{wLx^2}{2} - \frac{wL^2x}{2} + \frac{4wL^3}{81}\right]$$

Double Integration



$$y = \frac{1}{EI}\left[\frac{wLx^3}{18} - \frac{5wL^2x^2}{36} + C_2\right]$$

at $x = 0$, $y = 0$

therefore $C_2 = 0$

$$y = \frac{1}{EI}\left[\frac{wLx^3}{18} - \frac{5wL^2x^2}{36}\right]$$

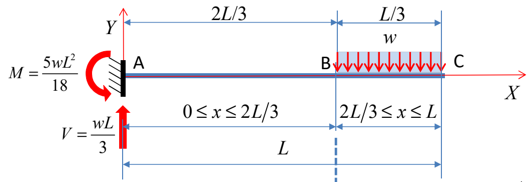
$$y = \frac{1}{EI}\left[-\frac{wx^4}{24} + \frac{wLx^3}{6} - \frac{wL^2x^2}{4} + \frac{4wL^3x}{81} + C_4\right]$$

at $x = 2L/3$, $y_{\text{Left}} = y_{\text{Right}}$

therefore $C_4 = -\frac{2wL^4}{243}$

$$y = \frac{1}{EI}\left[-\frac{wx^4}{24} + \frac{wLx^3}{6} - \frac{wL^2x^2}{4} + \frac{4wL^3x}{81} - \frac{2wL^4}{243}\right]$$

Double Integration



$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{wLx^2}{6} - \frac{5wL^2x}{18}\right]$$

$$y = \frac{1}{EI}\left[\frac{wLx^3}{18} - \frac{5wL^2x^2}{36}\right]$$

At point B, $x = 2L/3$

$$\left(\frac{dy}{dx}\right)_B = \frac{1}{EI}\left[\frac{wLx^2}{6} - \frac{5wL^2x}{18}\right] = \frac{1}{EI}\left[\frac{wL\left(\frac{2L}{3}\right)^2}{6} - \frac{5wL^2\left(\frac{2L}{3}\right)}{18}\right]$$

$$= -\frac{wL^3}{9EI}$$

$$y_B = \frac{1}{EI}\left[\frac{wLx^3}{18} - \frac{5wL^2x^2}{36}\right] = \frac{1}{EI}\left[\frac{wL\left(\frac{2L}{3}\right)^3}{18} - \frac{5wL^2\left(\frac{2L}{3}\right)^2}{36}\right]$$

$$= -\frac{11wL^4}{243EI}$$

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Double Integration

- Issues with the double integration approach
  - When the moment equation is continuous over the entire span
    - Must evaluate 2 constants of integration
    - Generally this evaluation is based on a slope and a deflection boundary condition or two deflection boundary conditions. Hopefully some of these at least will be zero.

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Double Integration

- Issues with the double integration approach
  - When there are discontinuities in load, intermediate supports, or other discontinuities, moment equations must be written for each continuous segment.
    - There will be two constants of integration for each segment.
    - In addition to boundary conditions there will be continuity conditions required to evaluate these constants. It becomes quite complex.

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Double Integration

- Issues with the double integration approach
  - There are other methods, built on the principles just discussed, that are often viewed as simpler when applied to general loading and continuity conditions.
  - One such method is moment-area which we will consider next.

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Moment-Area

- The two propositions that make up the moment-area method were introduced in 1873 by Charles Green.
- The method is based on the definition of the differential rotation at a particular location that was developed during our discussion of flexural strain energy in Lesson 2.

$$d\theta = \frac{\varepsilon_y}{y} = \frac{M_x}{EI}dx$$

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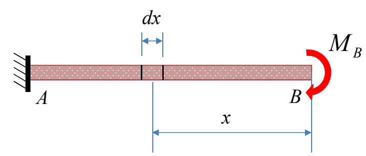
Moment-Area

- Proposition 1: *The difference in slope between any two sections of a loaded flexural member is equal to the area of the M/EI diagram between these two sections.*



Assume that the entire beam is rigid except for the segment over length dx

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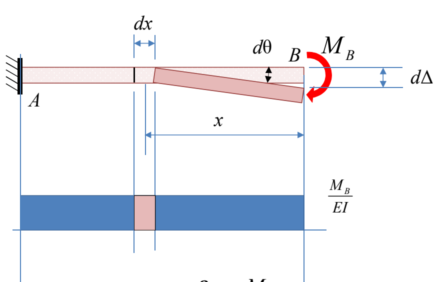
Moment-Area



We were just reminded that $d\theta = \frac{\varepsilon_y}{y} = \frac{M_x}{EI}dx$. It is seen that this is the area under the M/EI diagram over the length dx.

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Moment-Area

- Since the entire beam is elastic, each differential slice will contribute its own $d\theta$.
- Thus, to determine the difference in slope between any two points, A and B, we simply need to take the area of the M/EI diagram between points A and B.

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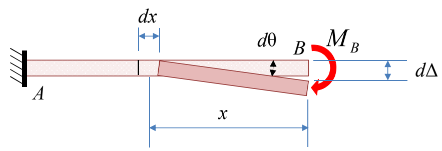
Moment-Area

- Proposition 2: *The tangential deviation of point B on a loaded flexural member, from a tangent to the deflection curve at point A, is equal to the moment of the area of the M/EI diagram between A and B about B.*



$$d\Delta = x\,d\theta$$

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Moment-Area

- Again, since the entire beam is elastic, each differential slice will contribute its own $d\Delta$.
- Thus, to determine the tangential deviation of point B from a tangent at A, we simply need to take the moment of the area of the M/EI diagram between points A and B about B.

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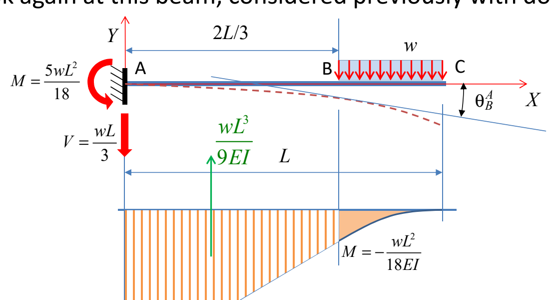
Moment-Area

Look again at this beam, considered previously with double integration.



$$\theta_B^A = \frac{1}{2}\left(-\frac{5wL^2}{18EI} - \frac{wL^2}{18EI}\right)\frac{2L}{3} = -\frac{wL^3}{9EI}$$

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(See slide 37)

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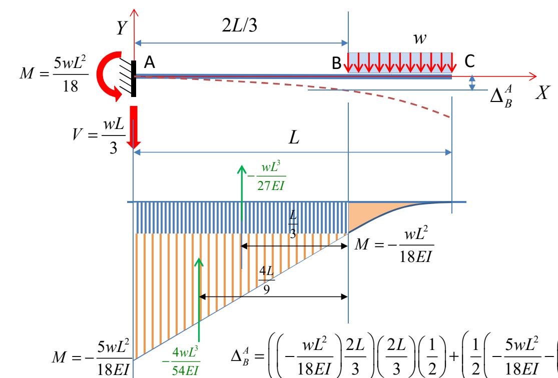
Moment-Area



$$\Delta_B^A = \left(\left(-\frac{wL^2}{18EI}\right)\frac{2L}{3}\right)\left(\frac{2L}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\left(-\frac{5wL^2}{18EI} - \left(-\frac{wL^2}{18EI}\right)\right)\frac{2L}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2L}{3}\right)$$

$$= -\frac{3wL^4}{243EI} - \frac{8wL^4}{243EI} = -\frac{11wL^4}{243EI}$$

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(See slide 37)

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Moment-Area

- Thoughts regarding the moment-area method
  - Slope can only be determined with respect to the slope at another point.
  - Deflection can only be determined from the tangent at another point (tangential deviation).
  - Method works best when there is a point of known slope (preferably zero).
  - Sign convention may also be treated intuitively within the context of a specific problem, knowing the sense of the rotations and deflections.

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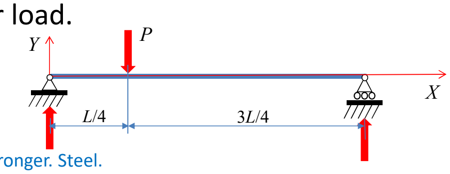
Moment-Area

- Thoughts regarding the moment-area method
  - When there is no point with a known slope, a reference slope must be determined.
  - Then all slopes and deflections can be determined with respect to that now known slope.
  - Consider the simply supported beam with an off center load.



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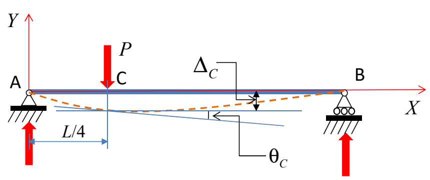
Moment-Area

- Determine the slope, θ_C, and deflection, Δ_C, at the location of the load, C.



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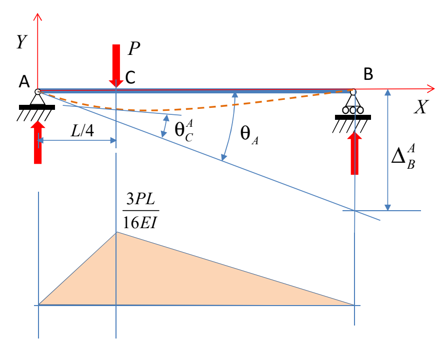
Moment-Area

Determine the slope at point C

1. The moment of the area between A and B about B = Δ_B^A
2. $\theta_A = \dfrac{\Delta_B^A}{L}$
3. The area between A and C = θ_C^A
4. $\theta_C = \theta_A - \theta_C^A$



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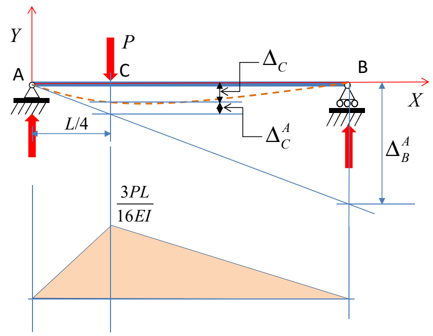
Moment-Area

Determine the deflection at point C

1. The moment of the area between A and B about B = Δ_B^A
2. The moment of the area between A and C about C = Δ_C^A
3. The distance from C on the undeformed beam to the slope at A = $\frac{\Delta_B^A}{4}$
4. $\Delta_C = \frac{\Delta_A^B}{4} - \Delta_C^A$



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Moment-Area

- Additional thoughts regarding the moment-area method.
  - Once the reference slope has been determined, all other slopes and deflections may be determined.
  - Each problem will require some thought before establishing a process toward solution.
  - An organized approach will be established through the use of the method of elastic weights, better known as conjugate beam.

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Moment-Area

- Question to consider
  - What are some of the drawbacks of the Moment Area Method? (Select all that apply.)
    a. Slope at a point is always first determined with respect to slope at another point
    b. Deflection at a point is only measured directly from slope at another point
    c. It only applies to beams with a point of zero slope
    d. It can not handle changes in moment of inertia
    e. It only applies to beams with a point of zero deflection

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Polling Question

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Elastic Weights

- The method of elastic weights was introduced in 1868 by Otto Mohr. It is simply another approach to expressing the moment-area method.
- Principle of Elastic Weights: *The slope and deflection at any point on a simply supported beam segment are given, respectively, by the shear and moment that results from applying the M/EI diagram as loading on an imaginary simply supported beam of the same length as the given beam.*

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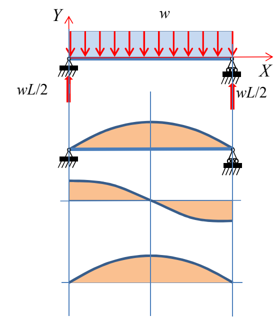
Elastic Weights



Load = w

$$\frac{d^2y}{dx^2} = \frac{M_x}{EI} = \frac{\int V_x dx}{EI} = \frac{\int\int w dx}{EI}$$

$$\frac{dy}{dx} = \int \frac{M_x}{EI} = \int \frac{\int V_x dx}{EI} = \int \frac{\int\int w dx}{EI}$$

$$y = \int\int \frac{M_x}{EI} = \int\int \frac{\int V_x dx}{EI} = \int\int \frac{\int\int w dx}{EI}$$

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Elastic Weights

- The analogy between the actual deflection line of a beam and the fictitious moment curve derived by loading the beam with the M/EI diagram holds for a simple beam only.
- To extend the principle of elastic weights to any beam, we must find a fictitious beam with supports that result in an appropriate set of boundary conditions, shears and moments on the conjugate beam, analogous to the corresponding slopes and deflections of the real beam.

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Conjugate Beam

- The conjugate beam theory is primarily due to Heinrich Müller-Breslau in 1885.
- Proposition 1: *The slope at any section of a loaded beam, relative to the original axis of the beam, is equal to the shear in the conjugate beam at the corresponding section.*
- Proposition 2: *The deflection at a given section of a loaded beam, relative to its original position, is equal to the bending moment at the corresponding section of the conjugate beam.*

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Conjugate Beam

- Given a beam and its M/EI diagram, find a substitute beam of the same span which, when loaded with the M/EI diagram, will exhibit a moment curve that is numerically equivalent to the deflection curve for the actual beam.

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Conjugate Beam

- The relationship between the real beam and the conjugate beam.
  - The conjugate beam is the same length as the real beam.
  - If there is slope on the real beam there must be shear on the conjugate beam.
  - If there is deflection on the real beam, there must be moment on the conjugate beam.

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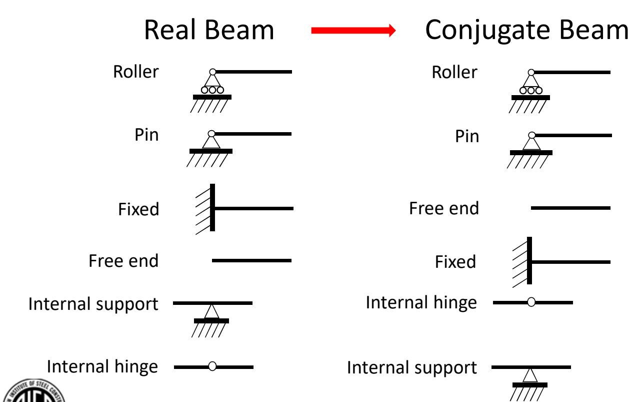
Conjugate Beam



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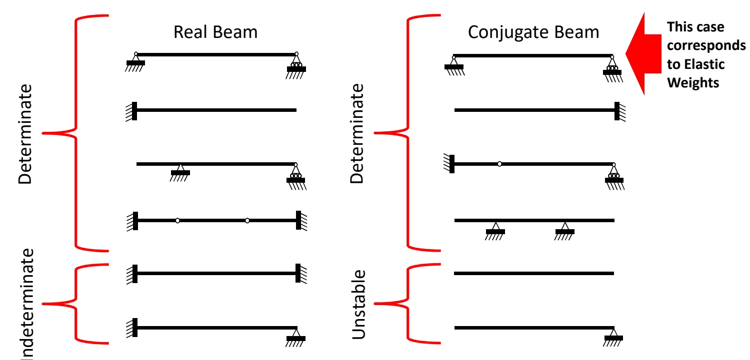
Conjugate Beam



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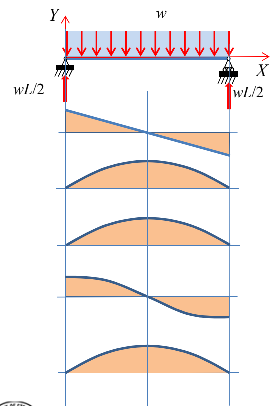
Elastic Weights



Load = w

$V_x = \int w dx$

$M_x = \int V_x dx = \int\int w dx$

$\frac{d^2y}{dx^2} = \frac{M_x}{EI} = \frac{\int V_x dx}{EI} = \frac{\int\int w dx}{EI}$

$\frac{dy}{dx} = \int \frac{M_x}{EI} = \int \frac{\int V_x dx}{EI} = \int \frac{\int\int w dx}{EI}$

$y = \int\int \frac{M_x}{EI} = \int\int \frac{\int V_x dx}{EI} = \int\int \frac{\int\int w dx}{EI}$

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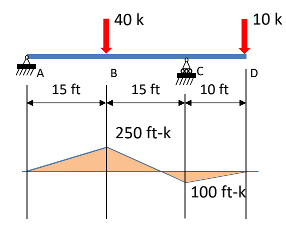
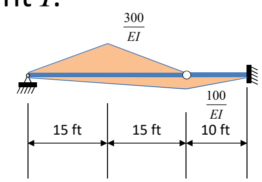
Conjugate Beam

- Determine the slopes and deflections at the critical points on the real beam with the loading shown and constant I.



Real Beam



Conjugate Beam

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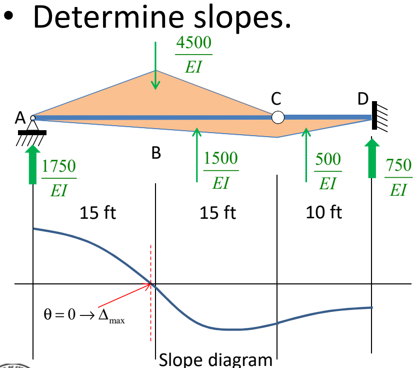
Conjugate Beam

- Determine slopes.



$V_A = \theta_A = \frac{1750}{EI}$

$V_B = \theta_B = \frac{-125}{EI}$

$V_C = \theta_C = \frac{-1250}{EI}$

$V_D = \theta_D = \frac{-750}{EI}$

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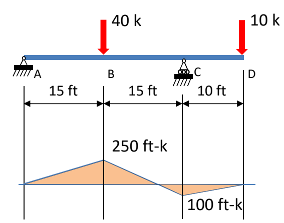
Conjugate Beam

- For $I = 500$ in.^{4}



$\theta_A = \frac{1750}{EI} = \frac{1750(144)}{29{,}000(500)} = 0.0174$ rad

$\theta_B = \frac{-125}{EI} = \frac{-125(144)}{29{,}000(500)} = -0.00124$ rad

$\theta_C = \frac{-1250}{EI} = \frac{-1250(144)}{29{,}000(500)} = -0.0124$ rad

$\theta_D = \frac{-750}{EI} = \frac{-750(144)}{29{,}000(500)} = -0.00745$ rad

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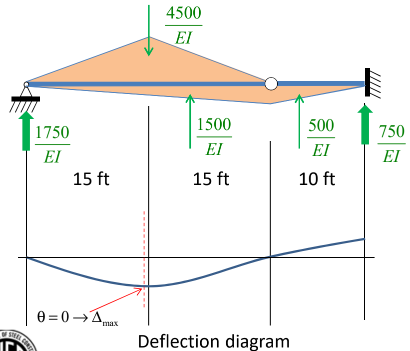
Conjugate Beam

- Determine deflections.



$$M_A = \Delta_A = 0$$

$$M_B = \Delta_B = \frac{-16,875}{EI}$$

$$M_C = \Delta_C = 0$$

$$M_D = \Delta_D = \frac{9167}{EI}$$

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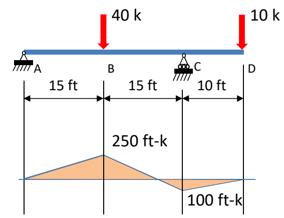
Conjugate Beam

- For $I = 500\ \text{in.}^4$



$$\Delta_A = 0$$

$$\Delta_B = \frac{-16,875}{EI} = \frac{-16,875(1728)}{29,000(500)} = -2.01\ \text{in.}$$

$$\Delta_C = 0$$

$$\Delta_D = \frac{9167}{EI} = \frac{9167(1728)}{29,000(500)} = 1.09\ \text{in.}$$

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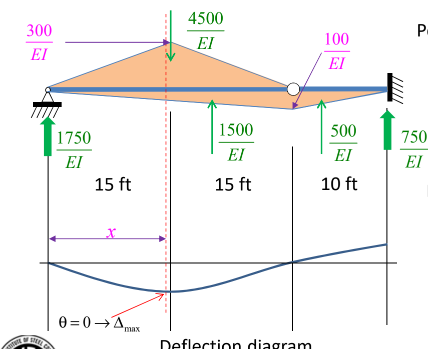
Conjugate Beam

- Determine the maximum deflection.



Point of zero shear on conjugate beam

$$\frac{1750}{EI} + \frac{100x}{30}\left(\frac{x}{2}\right) - \frac{300x}{15}\left(\frac{x}{2}\right) = 0$$

$$x = 14.49\ \text{ft}$$

Maximum moment on conjugate beam

$$M_{max} = \frac{1750}{EI}(14.49) + \frac{100}{30EI}\frac{(14.49)^2}{2}\left(\frac{14.49}{3}\right) - \frac{300}{15EI}\frac{(14.49)^2}{2}\left(\frac{14.49}{3}\right)$$

$$\Delta_{max} = -\frac{16,906}{EI}$$

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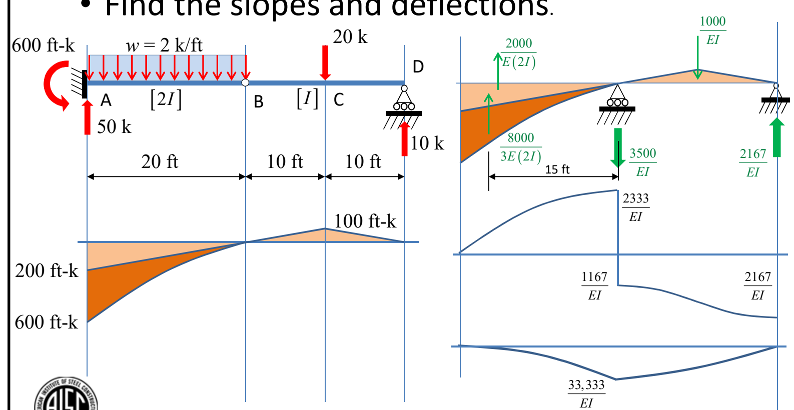
Conjugate Beam

- Find the slopes and deflections.



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Conjugate Beam

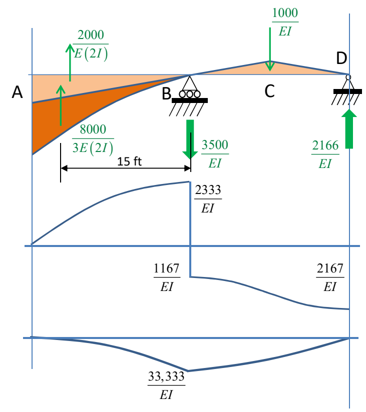
If $I = 1000$ in.^{4}

$$\theta_{B\text{-Left}} = \frac{2333(144)}{29{,}000(1000)} = 0.0116 \text{ rad}$$

$$\theta_{B\text{-Right}} = \frac{1167(144)}{29{,}000(1000)} = 0.00579 \text{ rad}$$

$$\theta_{D} = \frac{2167(144)}{29{,}000(1000)} = 0.0108 \text{ rad}$$

$$\Delta_{B} = \frac{33{,}333(1728)}{29{,}000(1000)} = 1.99 \text{ in.}$$



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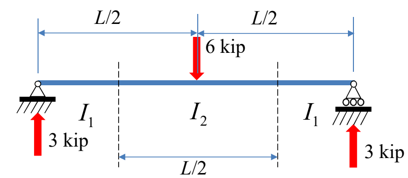
Conjugate Beam

- Consider a simple steel beam with a concentrated load at mid-span and the possibility of the middle half with a different moment of inertia.



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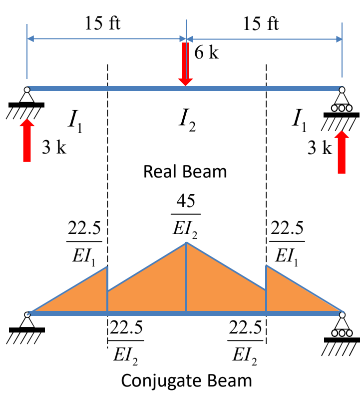
Conjugate Beam



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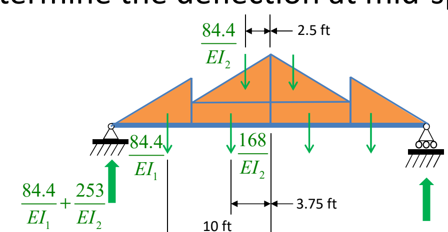
Conjugate Beam

- Determine the deflection at mid-span.



$$\Delta_{\text{mid-span}} = \left(\frac{84.4}{EI_1} + \frac{253}{EI_2}\right)(15) - \frac{84.4}{EI_1}(10) - \frac{168}{EI_2}(3.75) - \frac{84.4}{EI_2}(2.5)$$

$$\Delta_{\text{mid-span}} = \frac{422}{EI_1} + \frac{2954}{EI_2}$$

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Conjugate Beam

- Consider how the end and middle segments of this beam influence deflection based on the equation just developed.

$$\Delta_{\text{mid-span}} = \frac{422}{EI_1} + \frac{2954}{EI_2}$$

- If $I_1 = I_2$, what portion of the deflection is attributed to the center section?

$$\frac{\Delta_{\text{center}}}{\Delta_{\text{total}}} = \frac{\frac{2954}{EI}}{\frac{422}{EI} + \frac{2954}{EI}} = \frac{2954}{3376} = 0.875$$

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Conjugate Beam

- Questions to consider.
  - This beam deflects 2.0 in. when it has a uniform moment of inertia of 100 in.^{4}
    - Is it possible to reduce the deflection to 1.5 in. by increasing the moment of inertia of the end segments?
    - The end segments combine to have the same length as the mid-span segment. If you could only increase the moment of inertia of the ends or the mid-span, which would have the most impact on reducing the slope at the end of the beam?

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Polling Question

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Polling Question

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Conjugate Beam

- Although it may not always be the preferred method, it is particularly useful when the moment of inertia changes along the member.
- The conjugate beam method may also be used to determine the redundant moments on an indeterminate beam.
- It is also very well suited for determining fixed end moments which will be useful with methods to be introduced later.

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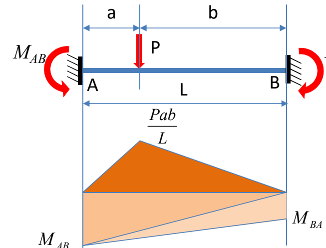
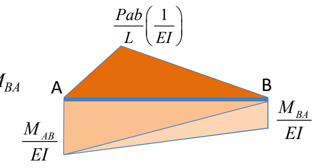
Conjugate Beam

- Determine a general equation for the fixed end moments on a beam with a concentrated load as shown.



Real Beam



Conjugate Beam

Note: the real beam is indeterminate so the conjugate beam is unstable.

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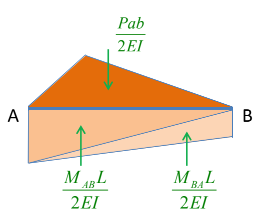
Conjugate Beam



To determine the unknown end moments, M_{AB} and M_{BA}, apply the equations of equilibrium.

$$\sum F_y = 0$$

$$\sum M_A = 0$$

$$\sum F_y = 0 = -\frac{Pab}{2EI} + \frac{M_{AB}L}{2EI} + \frac{M_{BA}L}{2EI}$$

$$M_{BA} = \frac{Pab}{L} - M_{AB}$$

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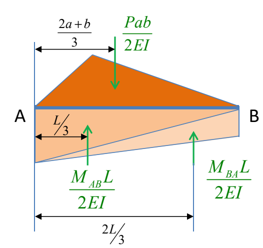
Conjugate Beam



$$\sum M_A = 0 = \left(\frac{Pab}{2EI}\right)\left(\frac{2a+b}{3}\right) - \frac{M_{AB}L}{2EI}\left(\frac{L}{3}\right) - \frac{M_{BA}L}{2EI}\left(\frac{2L}{3}\right)$$

Substituting for M_{BA} and rearranging

$$0 = \left(\frac{Pab}{2EI}\right)\left(\frac{2a+b}{3}\right) - \frac{M_{AB}L}{2EI}\left(\frac{L}{3}\right) - \left[\frac{Pab}{L} - M_{AB}\right]\left(\frac{L}{2EI}\right)\left(\frac{2L}{3}\right)$$

$$M_{AB} = \frac{Pab^2}{L^2}$$

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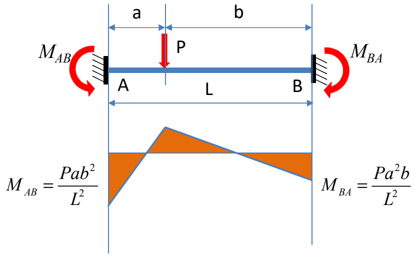
Conjugate Beam

- Substitute M_{AB} into M_{BA} equation and solve for M_{BA}.



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Summary

- Developed the basic beam theory that permitted the use of double integration for slope and deflection calculations
- Extended the basic beam theory to the moment area method for determining slopes and deflections.
- Introduced the limited method of elastic weights.
- Developed the more useful conjugate beam method.
- Considered the influence of changing moment of inertia on beam behavior.
- Illustrated fixed end moment determination by the conjugate beam method.

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Lesson 5

- Indeterminate Structures and the General Method
  - Extend the conjugate beam method for use in determining redundant moments on indeterminate beams.
  - Develop the method of consistent deformations (the General Method).
  - Analyze indeterminate structures by consistent deformation including axial, flexural and shearing deformations.
  - Consider the AISC 360-16 requirements for analysis.

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Within 2 business days…

- You will receive an email on how to report attendance from: registration@aisc.org.
- Be on the lookout: Check your spam filter! Check your junk folder!
- Completely fill out online form. Don't forget to check the boxes next to each attendee's name!

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Within 2 business days…

- New reporting site (URL will be provided in the forthcoming email).
- Username: Same as AISC website username.
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One certificate will be issued at the conclusion of all 8 sessions.

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8-Session Registrants

Access to the quiz: Information for accessing the quiz will be emailed to you by Wednesday. It will contain a link to access the quiz. EMAIL COMES FROM NIGHTSCHOOL@AISC.ORG

Quiz and Attendance records: Posted Tuesday mornings.
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Reasons for quiz:

•EEU – must take all quizzes and final to receive EEU
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•REINFORCEMENT – Reinforce what you learned tonight. Get more out of the course.

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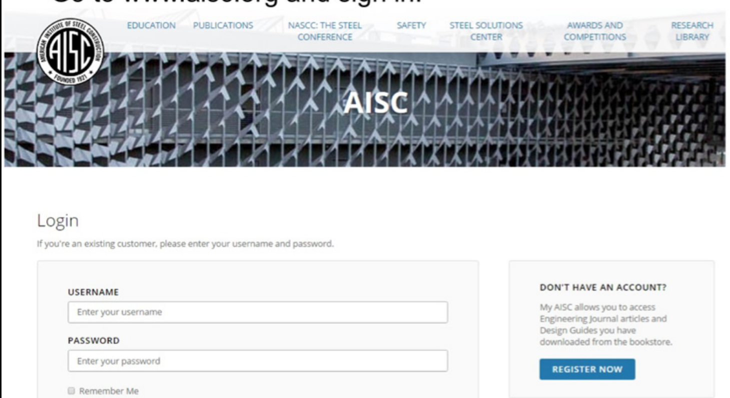

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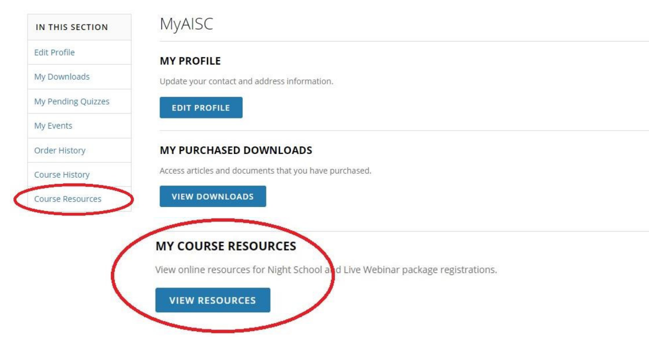

Please fill out our brief survey at the conclusion of the webinar. We greatly appreciate your feedback.

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